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# Simple Zeros of the Riemann Zeta-Function

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By estimating the change in argument of a certain function it has been shown that at least 0.3474 of the nonreal zeros of $\zeta(s)$ are simple. It is shown here that a more general function containing a real parameter can be used. An optimal choice of which gives a proportion greater than 0.3532.

In 1974, Levinson developed a new method for estimating the number of zeros of the Riemann zeta-function on the critical line. Calculations described in [2] led to

$$N_0(T) > 0.3474N(T), \tag{1}$$

where $N(T)$ is the number of zeros of $\zeta(s)$ with $0 < \sigma < 1$, $0 < t < T$ and $N_0(T)$ is the number of these with $\sigma = \frac{1}{2}$. Heath-Brown [1] has observed that (1) holds with $N_0(T)$ replaced by $N_1(T)$, the number of simple zeros of $\zeta(\frac{1}{2} + it)$ in $(0, T)$. The author has been told by Conrey that he has proved $N_0(T) > 0.3658N(T)$ and $N_1(T) > 0.3485N(T)$.

The object of this note is to indicate how the latter estimate can be refined by a small modification of Levinson's proof of (1). The following result will be sketched:

THEOREM. *If $T$ is sufficiently large,*

$$N_1(T) > 0.3532N(T).$$

*This will be proved in $(T, T+U)$, where $U = T(\log T/2\pi)^{-10}$.*

The proof of (1) depended on estimating the change in the argument of $\zeta(s) - (\chi/\chi')(s)\,\zeta'(s)$, where $\zeta(s) = \chi(s)\,\zeta(1-s)$. It is more efficient to consider $\zeta(s) - \alpha(\chi/\chi')(s)\,\zeta'(s)$, where $\alpha$ is real and to choose $\alpha$ optimally.



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The symmetric form of the functional equation is $h(s)\,\zeta(s) = h(1-s)\,\zeta(1-s)$ with $h(s) = \pi^{-s/2}\Gamma(\frac{1}{2}s)$. Differentiation gives

$$h(s)\,\zeta'(s) + h(1-s)\,\zeta'(1-s) = -h'(s)\,\zeta(s) - h'(1-s)\,\zeta(1-s)$$
$$= \frac{\chi'}{\chi}(s)\,h(s)\,\zeta(s). \tag{2}$$

If (2) is multiplied by $-a(\chi/\chi')(s)$ and the equation

$$h(s)\,\zeta(s) + h(1-s)\,\zeta(1-s) = 2h(s)\,\zeta(s)$$

is added to the result, the identity

$$h(s)\,G(s) + h(1-s)\,G(1-s) = (2-a)\,h(s)\,\zeta(s) \tag{3}$$

is obtained. Here

$$G(s) = \zeta(s) - a\,\frac{\chi}{\chi'}(s)\,\zeta'(s).$$

Equation (3) will lead to a lower bound for $N_1(T)$. A proof of the theorem could also begin from the fact that if $0 < a < 2$, $\zeta(s)$ and $G(s)$ have essentially the same number of zeros in the right side of the critical strip. Actually, instead of $G(s)$, the simpler function

$$G_0(s) = \zeta(s) + aL^{-1}\zeta'(s)$$

can be used, where $L = \log(T/2\pi)$. The identity takes the form

$$h(s)\,G_0(s) + h(1-s)\,G_0(1-s) = (2 + aL^{-1}(\chi'/\chi)(s))\,h(s)\,\zeta(s). \tag{4}$$

Since $(\chi'/\chi)(\frac{1}{2} + it)$ is decreasing for $t > t_0$, the zeros of the function on the right side of (4) with $\sigma = \frac{1}{2}$ are those of $\zeta(s)$ itself with one possible exception. If $a$ is real, the left side is $2\,\mathrm{Re}\,h(s)\,G_0(s)$ so zeros of $\zeta(s)$ with $\sigma = \frac{1}{2}$ coincide with those of $\mathrm{Re}\,h(s)\,G_0(s)$. A zero of $\zeta(s)$ which is not a zero of $G_0(s)$ is simple. Much as in [1] one obtains the inequality

$$N_1(T+U) - N_1(T) \geqslant \frac{1}{\pi}\arg h(s)\,G_0(s) - 1 + o(1), \tag{5}$$

where arg is the change in argument on a contour from $\frac{1}{2} + iT$ to $\frac{1}{2} + i(T+U)$ formed by identing the $\frac{1}{2}$-line between these points with small left semicircles centered at zeros of $G_0(s)$.

The right side of (5) is

$$\frac{UL}{2\pi} - 2N(G_0) + O(U), \tag{6}$$

where $N(G_0)$ is the number of zeros of $G_0(s)$ with $\frac{1}{2} < \sigma < 3$, $T < t < T + U$ if the semicircles are small enough.

As in [4] a mollifier is needed to get a usable bound for $N(G_0)$. Thus $N(\psi G_0)$ will be estimated, where

$$\psi(s) = \sum_{j \leqslant y} b(j) j^{-s},$$

and $y = T^{1/2} L^{-20}$. The choice for $b(j)$ is

$$b(j) = \frac{\mu(j)}{j^{1-2a}} \frac{y^{1-2a} - j^{1-2a}}{y^{1-2a} - 1}. \tag{7}$$

Levinson obtained (1) by using this $b(j)$ with $\alpha = 1$. It turns out that for this $b(j)$ the optimal value for $\alpha$ is about 1.0355.

An upper bound for $N(\psi G_0)$ may be deduced from Littlewood's lemma. Much as in [4, p. 386] this leads to

$$2\pi(\tfrac{1}{2} - a) N(\psi G_0) \leqslant \int_T^{T+U} \log |\psi G_0(a + it)| \, dt + O(UL^{-1}), \tag{8}$$

where $a = \frac{1}{2} - \lambda L^{-1}$ and $\lambda$ is a positive number that will be chosen later.

Use of the Riemann–Siegel formula as in [4, p. 389] and $(\chi'/\chi)(s) = -L + O(L^{-10})$ for $T < t < T + U$ shows that $G_0(s)$ is essentially

$$\sum_{n \leqslant X} P\left(\frac{\log n}{L}\right) n^{-s} + \chi(s) \sum_{n \leqslant X} P\left(1 - \frac{\log n}{L}\right) n^{s-1}, \tag{9}$$

where $P(x) = 1 - \alpha x$ and $X = (t/2\pi)^{1/2}$.

By Jensen's inequality the integral in (8) is at most

$$U \log \left(\frac{1}{U} \int_T^{T+U} |\psi G_0(a + it)| \, dt\right).$$

If the expression in (9) is denoted $H_0(s)$, it is found that

$$\begin{aligned} \int_T^{T+U} |\psi G_0(a + it)| \, dt &\leqslant \int_T^{T+U} |\psi H_0(a + it)| \, dt + O(UL^{-1}) \\ &\leqslant U^{1/2} J^{1/2} + O(UL^{-1}), \end{aligned}$$

where

$$J = \int_T^{T+U} |\psi H_0(a + it)|^2 \, dt.$$

It follows that

$$2\pi(\tfrac{1}{2}-a)\,N(\psi G_0) \leqslant U\log(U^{-1/2}J^{1/2}+O(L^{-1}))+O(UL^{-1})$$

so by (5) and (6)

$$\begin{aligned} &N_1(T+U)-N_1(T) \\ &\qquad \geqslant \frac{UL}{2\pi}-\frac{UL}{\lambda\pi}\log(U^{-1/2}J^{1/2}+O(L^{-1}))+O(U). \end{aligned} \tag{10}$$

To prove the theorem it is necessary to compute $J$ which involves calculations that are very similar to those in [3, 4]. For $i=0, 1, 2$, let $S_i$ and $K_i$ denote the sums in [4, p. 422]. Ultimately the methods of [3, 4] lead to

$$J=(A\alpha^2+B\alpha+C)\,U+O(UL^{-5}), \tag{11}$$

where

$$\begin{aligned} A=&\frac{-2}{(1-2a)^3L^2}S_0+\frac{2}{(1-2a)^2L^2}S_1-\frac{1}{(1-2a)L^2}S_2 \\ &+\tau^{2-4a}\left[\frac{1}{1-2a}-\frac{2}{(1-2a)^2L}+\frac{2}{(1-2a)^3L^2}\right]K_0 \\ &+\tau^{2-4a}\left[\frac{-2}{(1-2a)L}+\frac{2}{(1-2a)^2L^2}\right]K_1 \\ &+\tau^{2-4a}\left[\frac{1}{(1-2a)L^2}\right]K_2, \end{aligned} \tag{12}$$

$$\begin{aligned} B=&\frac{-2}{(1-2a)^2L}S_0+\frac{2}{(1-2a)L}S_1 \\ &+\tau^{2-4a}\left[\frac{-2}{1-2a}+\frac{2}{(1-2a)^2L}\right]K_0 \\ &+\tau^{2-4a}\left[\frac{2}{(1-2a)L}\right]K_1, \end{aligned} \tag{13}$$

$$C=\frac{-1}{1-2a}S_0+\frac{\tau^{2-4a}}{1-2a}K_0, \tag{14}$$

and $\tau=(T/2\pi)^{1/2}$.

Equations (12)–(14) were derived for arbitrary $b(j)$ subject to $b(1)=1$ and $|b(j)|\leqslant 1$. For $b(j)$ as defined by (7) formulas for $S_i$ were found by Levinson. Since $y^{1-2a}$ is essentially $e^{\lambda}$ the formulas in [2, p. 296] are

$$S_0 = \frac{2\lambda}{e^{\lambda} - 1} \frac{1}{L} + O(L^{-2} \log^5 L), \tag{15}$$

$$S_1 = \frac{-\lambda e^{\lambda}}{(e^{\lambda} - 1)^2} + \frac{1}{e^{\lambda} - 1} + O(L^{-1} \log^5 L), \tag{16}$$

$$S_2 = \frac{e^{2\lambda} - 2\lambda e^{\lambda} - 1}{2\lambda (e^{\lambda} - 1)^2} L + O(\log^5 L). \tag{17}$$

The results for $K_i$ are

$$K_0 = \frac{2\lambda e^{\lambda}}{e^{\lambda} - 1} \frac{1}{L} + O(L^{-2} \log^5 L), \tag{18}$$

$$K_1 = \frac{\lambda e^{\lambda}}{(e^{\lambda} - 1)^2} - \frac{e^{\lambda}}{e^{\lambda} - 1} + O(L^{-1} \log^5 L), \tag{19}$$

$$K_2 = \frac{e^{2\lambda} - 2\lambda e^{\lambda} - 1}{2\lambda (e^{\lambda} - 1)^2} L + O(\log^5 L), \tag{20}$$

which can be derived in a very similar way from Eqs. (4)–(6) of [2]. Using Eqs. (15)–(20) and $1 - 2a = 2\lambda L^{-1}$, $A$, $B$, and $C$ may be expressed as functions of $\lambda$. It is found after simplifying that

$$A = \frac{4\lambda^2 e^{4\lambda} + e^{4\lambda} - 8\lambda^2 e^{3\lambda} - 2e^{2\lambda} + 1}{4\lambda^2 (e^{\lambda} - 1)^2}, \tag{21}$$

$$B = \frac{-2e^{4\lambda} + 3e^{3\lambda} - e^{\lambda}}{(e^{\lambda} - 1)^2}, \tag{22}$$

$$C = \frac{e^{4\lambda} - e^{3\lambda} - e^{\lambda} + 1}{(e^{\lambda} - 1)^2}, \tag{23}$$

apart from terms that are $O(L^{-1} \log^5 L)$.

*Proof of the Theorem.* Adding the last three equations gives

$$\frac{(e^{\lambda} + 1)^2}{4\lambda^2} - \frac{2e^{\lambda} - 1}{(e^{\lambda} - 1)^2} + O(L^{-1} \log^5 L)$$

as the value of $U^{-1}J$ when $\alpha = 1$. This agrees with [2, p. 297]. However, to make the polynomial in (11) as small as possible one should let $\alpha = -B/2A$, where its value is $C - B^2/4A$. Hence

$$J = \left(C - \frac{B^2}{4A}\right) U + O(UL^{-5})$$

for this $\alpha$. By (10),

$$N_1(T+U) - N_1(T) \geqslant \frac{UL}{2\pi}\left[1 - \frac{1}{\lambda}\log\left(C - \frac{B^2}{4A}\right)\right] + O(U). \qquad (24)$$

For $\lambda = 1.26$ easy calculations using (21)–(23) give the approximations

$$A = 13.70476009,$$
$$B = -28.38261482,$$
$$C = 16.95401816.$$

Thus $\log(C - (B^2/4A)) < 0.8149$ and the theorem follows from (24). Here $\alpha$ is about 1.0355.

Equation (3) is the first of an infinite sequence of identities for $\zeta(s)$. Let $F_0(s) = \zeta(s)$ and for $j \geqslant 1$ let $F_j(s) = F_{j-1}(s) - 2(\chi/\chi')(s)\,F'_{j-1}(s)$. Equation (3) with $\alpha = 2$ gives

$$h(s)\,F_1(s) + h(1-s)\,F_1(1-s) = 0,$$

since $F_1(s) = G(s)$. If

$$h(s)\,F_{2j-1}(s) + h(1-s)\,F_{2j-1}(1-s) = 0, \qquad (25)$$

then differentiation gives

$$h(s)\,F'_{2j-1}(s) - h(1-s)\,F'_{2j-1}(1-s) = \frac{\chi'}{\chi}(s)\,h(s)\,F_{2j-1}(s).$$

By multiplying by $-2(\chi/\chi')(s)$ and using (25) one obtains

$$h(s)\,F_{2j}(s) - h(1-s)\,F_{2j}(1-s) = 0. \qquad (26)$$

Differentiating, multiplying by $-2(\chi/\chi')(s)$ and using (26) leads to

$$h(s)\,F_{2j+1}(s) + h(1-s)\,F_{2j+1}(1-s) = 0,$$

so (25) holds for all $j \geqslant 1$ by induction. By (25),

$$h(s)[\alpha_{2j-1}(F_{2j-1}(s) - \zeta(s))] + h(1-s)[\alpha_{2j-1}(F_{2j-1}(1-s) - \zeta(1-s))]$$
$$= -2\alpha_{2j-1}h(s)\,\zeta(s),$$

where $\alpha_{2j-1}$ is real. Therefore

$$h(s)\,G(s) + h(1-s)\,G(1-s) = 2h(s)\,\zeta(s)\left(1 - \sum_{j=1}^{N} \alpha_{2j-1}\right), \qquad (27)$$

where

$$G(s) = \zeta(s) + \sum_{j=1}^{N} \alpha_{2j-1}(F_{2j-1}(s) - \zeta(s)).$$

Equation (3) is the case $N = 1$. There $\alpha = 2\alpha_1$. Calculations for the case $N = 2$ indicate that $N_0(T) > 0.36N(T)$ is obtained using the mollifier defined by (7) with $\lambda = 1.3$, $\alpha_1 = 1.028$, $\alpha_3 = -.35$.

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